

GAUSS-NEWTON METHOD FOR PHASE RETRIEVAL

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ABSTRACT. In this paper, we develop a concrete algorithm for phase retrieval, which we refer to as Gauss-Newton algorithm. In short, this algorithm starts with a good initial estimation, which is obtained by a modified spectral method, and then update the iteration point by a Gauss-Newton iteration step. We prove that a re-sampled version of this algorithm quadratically converges to the solution for the real case with the number of random measurements being nearly minimal. Numerical experiments also show that Gauss-Newton method has better performance over the other algorithms.

1. INTRODUCTION

1.1. Phase Retrieval. Recovering a signal from the magnitude of measurements, known as *phase retrieval*, has important applications in X-ray imaging, crystallography, electron microscopy and coherence theory. Suppose that $\{a_1, \dots, a_N\} \subset \mathbb{H}^n$ is a frame, i.e., $\text{span}\{a_1, \dots, a_N\} = \mathbb{H}^n$. The phase retrieval problem can be formulated in the form of solving quadratic equations:

$$(1.1) \quad y_j = |\langle a_j, z \rangle|^2, \quad j = 1, 2, \dots, m,$$

where $a_j \in \mathbb{H}^n$ ($\mathbb{H} = \mathbb{C}$ or \mathbb{R}) are the sensing vectors and $z \in \mathbb{H}^n$ is the desired variable. Throughout this paper, we use $A := [a_1, \dots, a_m]^* \in \mathbb{H}^{m \times n}$ to denote the measurement matrix.

Recently, phase retrieval attracts much attention [1–3] and many algorithms are developed for solving it. A well-known method is the error reduction algorithm [9, 10]. Despite the algorithm is used in many applications, there are few theoretical results about the global convergence property of it. In [12], a re-sampled version of the error reduction algorithm, the alternating minimization algorithm, is introduced with proving that the algorithm geometrically converges to the true signal up to an accuracy of ϵ provided the measurement matrix $A \in \mathbb{R}^{m \times n}$ is Gaussian random matrix with $m = O(n \log^3 n \log \frac{1}{\epsilon})$. In fact, to attain the accuracy of ϵ , the algorithm needs $O(\log \frac{1}{\epsilon})$ iterations and different measurements are employed in each iteration of the algorithm. Wirtinger flow (WF) method is first introduced to solve the phase retrieval problem in [7]. WF method combines a good initial guess, which is obtained by spectral method, and a series of updates which refine the initial estimate by a deformation of the gradient descent method. It is proved that WF algorithm converges to an exact solution on a linear rate from $O(n \log n)$ Gaussian random measurements [7]. In fact, it is shown in [7] that

$$\text{dist}(x_{k+1} - z) \leq \rho \cdot \text{dist}(x_k - z),$$

where x_k is the output of the k -th iteration of WF method, z is the true signal, $0 < \rho < 1$ and the definition of $\text{dist}(\cdot)$ is given in Section 1.3. The truncated WF method is introduced in [4], which improves the performance of WF method with showing that $O(n)$ Gaussian random measurements are enough to attain the linear convergence rate. Despite many iterative algorithms to solve phase retrieval, a recent approach for phase retrieval is to recast it as a semi-definite programming (SDP), such as PhaseLift [5, 8]. The PhaseLift is to lift a vector problem to a rank-1 matrix one and then one can recover the rank-1 matrix by minimizing the trace of matrices. Though one can prove that PhaseLift can provide the exact solution using $O(n)$ measurements, the computational cost is large when the dimension of the signal is high.

In many applications, the signals to be reconstructed are sparse. Thus it is natural to develop algorithms to recover sparse signals from the magnitude of measurements, which is also known as sparse phase retrieval problem. The ℓ_1 model for the recovery of sparse signals from the magnitude of measurements is studied

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in [11, 15, 16]. A greedy algorithm GESPAR for solving sparse phase retrieval is presented in [13]. The core step of the method is to use the damp Gauss-Newton method to solve a non-linear least square problem. They choose the step size by backtracking and prove that damp Gauss-Newton method converges to a stationary point. In [6], one investigates the performance of WF method for the recovery of real sparse signals from the phaseless measurement.

1.2. Our contribution. The aim of this paper is twofold. We first present an alternative initial guess which is the eigenvector corresponding to the minimum eigenvalue of $\frac{1}{m} \sum_{r=1}^m \exp(-y_r) a_r a_r^*$. Compared with the one obtained by the spectral method, the new initial guess can reach accuracy with $O(n)$ Gaussian random measurements while the spectral method requires $O(n \log n)$. The numerical experiments also show that the new method has a better performance. Our second aim is to set up a new algorithm for solving phase retrieval problem. In the algorithm, starting with our initial guess, we refine the initial estimation by iteratively applying an update rule, which comes from a Gauss-Newton iteration. Thus for the convenience of description, we name this algorithm as Gauss-Newton algorithm. We investigate the performance of the Gauss-Newton algorithm with showing that a re-sampled version of this Gauss-Newton algorithm can quadratically converge to the true signal up to a global sign for the real case, i.e.,

$$\text{dist}(x_{k+1}, z) \leq \rho \cdot (\text{dist}(x_k, z))^2,$$

where x_k is the output of the k -th iteration and $0 < \rho < 1$. Hence, Gauss-Newton method with a re-sampled version has a quadratic convergence rate. This implies that, to reach the accuracy ϵ , Gauss-Newton method needs $\mathcal{O}(\log \log \frac{1}{\epsilon})$ iterations which has an improvement over the previous algorithms.

1.3. Notations. Throughout the paper, we reserve C , c and γ , and their indexed versions to denote positive constants. Their value vary with the context. We use $z \in \mathbb{H}^n$ to denote the exact signal we want to recover. Without loss of generality and to simplify exposition, we shall assume $\|z\| = 1$. Throughout this paper, when no subscript is used, $\|\cdot\|$ denotes the Euclidian norm, i.e., $\|\cdot\| = \|\cdot\|_2$. We use the Gaussian random vectors $a_j \in \mathbb{H}^n$, $j = 1, \dots, m$ as the sampling vectors and obtain $y_j = |\langle a_j, z \rangle|^2$, $j = 1, \dots, m$. Here we say the sampling vectors are the Gaussian random measurements if $a_j \in \mathbb{C}^n$, $j = 1, \dots, m$ are i.i.d. $\mathcal{N}(0, I/2) + i\mathcal{N}(0, I/2)$ random variables or $a_j \in \mathbb{R}^n$, $j = 1, \dots, m$ are i.i.d. $\mathcal{N}(0, I)$ random variables. Denote x_k as the k -th iteration point and S_k , $k = 0, 1, \dots, k_0$ as the line segment between x_k and z , i.e.,

$$S_k := \{\lambda z + (1 - \lambda)x_k : 0 \leq \lambda \leq 1\}.$$

Then we give the definition of distance of $x \in \mathbb{H}^n$ to the solution set as follows.

$$\text{dist}(x, z) = \begin{cases} \min_{\phi \in [0, 2\pi)} \|z - e^{i\phi} x\| & \mathbb{H} = \mathbb{C}, \\ \min\{\|z - x\|, \|z + x\|\} & \mathbb{H} = \mathbb{R}. \end{cases}$$

1.4. Organization. The paper is organized as follows. In Section 2, we introduce a modified spectral method and prove that it can provide a good initial guess by only $O(n)$ Gaussian random measurements. The Gauss-Newton algorithm for real phase retrieval is given in Section 3 and we prove that a re-sampled version of this method can achieve quadratic convergence. Some numerical experiments are given in the last section.

2. INITIALIZATION

2.1. Initialization method. The first step of Gauss-Newton method is to choose an initial estimation. So far, one of popular methods for initialization is to choose the leading eigenvector of $\frac{1}{m} \sum_{r=1}^m y_r a_r a_r^*$ as the initial guess [7, 9, 10]. In fact, when a_r are the Gaussian random measurements, we have

$$\mathbb{E}\left(\frac{1}{m} \sum_{r=1}^m y_r a_r a_r^*\right) = I_n + 2zz^*$$

and any leading eigenvector of $I_n + 2zz^*$ is of the form cz for a constant c . In [7], Candès, Li and Soltanolkotabi proved that when $m \geq c_0 n \log n$ and a_r , $r = 1, \dots, m$ are Gaussian random measurements,

$$\text{dist}(z_0, z) \leq \frac{1}{8}$$

holds with probability at least $1 - 10 \exp(-\gamma n) - 8/n^2$ (see also [12]). Here, z_0 is the eigenvector corresponding to the largest eigenvalue of $\frac{1}{m} \sum_{r=1}^m y_r a_r a_r^*$. A modified spectral method is introduced in [4], which precludes y_r with large magnitudes. Particularly, they select the initial value as the leading eigenvector of

$$\frac{1}{m} \sum_{r=1}^m y_r a_r a_r^* I_{\{|y_r| \leq \beta_y \lambda^2\}},$$

where β_y is an appropriate truncation criteria and $\lambda^2 = \frac{\sum_r y_r}{m}$. This method only requires the number of measurements is on the same order of unknowns. To state conveniently, we name the first method as SI (Spectral Initialization) and the second method as TSI (Truncated Spectral Initialization).

In this section, we introduce a new method for initialization, which is stated in Algorithm 1. In fact, the initial guess is chosen as the eigenvector corresponding to the minimum eigenvalue of

$$Y := \frac{1}{m} \sum_{r=1}^m \exp(-y_r) a_r a_r^*.$$

The new method can obtain an alternative initial value by nearly optimal number of measurements (see Theorem 2.1). Beyond theoretical results, numerical experiments also show that this method has better performance than that of SI and TSI (see Example 4.1).

Algorithm 1 Initialization

Input: Observations $y \in \mathbb{R}^m$

Set

$$\lambda^2 = \frac{\sum_r y_r}{m}.$$

Set x_0 , normalized to $\|x_0\|_2 = \lambda$, to be the eigenvector corresponding to the minimum eigenvalue of

$$Y = \frac{1}{m} \sum_{r=1}^m \exp(-y_r) a_r a_r^*.$$

Output: Initial guess x_0 .

2.2. The performance of Algorithm 1. The following theorem shows the performance of Algorithm 1. Here we suppose $\mathbb{H} = \mathbb{C}$ and prove that the initial guess x_0 is not far from cz , $|c| = 1$.

Theorem 2.1. *Suppose that $z \in \mathbb{C}^n$ with $\|z\| = 1$ and x_0 is the output of Algorithm 1. For any $0 < \theta \leq 1/2$, when $m \geq Cn$,*

$$\text{dist}(x_0, z) = \min_{\phi \in [0, 2\pi)} \|z - e^{i\phi} x_0\| \leq 2\sqrt{5\theta}$$

holds with probability at least $1 - 9 \exp(-\gamma_\theta n)$, where $\gamma_\theta > 0$ is a constant depending on θ and C is a constant depending on θ .

To prove this theorem, we first recall some useful results.

Theorem 2.2 (Wely Theorem). *Suppose $A, B \in \mathbb{C}^{n \times n}$ are two Hermite matrices. The eigenvalues of A are denoted as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and the eigenvalues of B are denoted as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. Then we have*

$$|\mu_i - \lambda_i| \leq \|A - B\|_2, \quad i = 1, 2, \dots, n.$$

Lemma 2.1. [Theorem 5.39 in [14]] *Assume the sampling vectors $a_j \in \mathbb{C}^n$ are the Gaussian random measurements. For any $\eta > 0$, when the number of samples obeys $m \geq c_\eta \cdot n$,*

$$(2.2) \quad \|I_n - \frac{1}{m} \sum_{j=1}^m a_j a_j^*\| \leq \eta$$

holds with probability at least $1 - 2 \exp(-\gamma m)$. On this event, we have

$$(1 - \eta) \|x\|^2 \leq \frac{1}{m} \sum_{j=1}^m |a_j^* x|^2 \leq (1 + \eta) \|x\|^2, \quad \text{for all } x \in \mathbb{C}^n.$$

The next lemma plays an essential role in proving the Theorem 2.1.

Lemma 2.2. *Let $z \in \mathbb{C}^n$ with $\|z\| = 1$ be a fixed vector. Suppose $a_j \in \mathbb{C}^n, j = 1, 2, \dots, m$ are the Gaussian random measurements. Set*

$$Y := \frac{1}{m} \sum_{j=1}^m \exp(-|a_j^* z|^2) a_j a_j^*.$$

Then for any $\theta > 0$,

$$\|Y - \mathbb{E}Y\| \leq \frac{\theta}{4}$$

holds with probability at least $1 - 7 \exp(-\gamma'_\theta n)$ provided $m \geq C'n$, where $\gamma'_\theta > 0$ is a constant depending on θ and C' is an absolute constant.

Proof. As $a_j \in \mathbb{C}^n, j = 1, 2, \dots, m$ are the Gaussian random measurements, a simple moment calculation gives

$$\mathbb{E}Y = \frac{1}{2}(I_n - \frac{1}{2}zz^*).$$

By unitary invariance, we can take $z = e_1 = (1, 0, \dots, 0) \in \mathbb{C}^n$. To this end, it is enough to prove that

$$(2.3) \quad \left\| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j a_j^* - \frac{1}{2}(I_n - \frac{1}{2}e_1 e_1^*) \right\| \leq \frac{\theta}{4}.$$

We use $u(1)$ to denote the first coordinate of a vector u . Then (2.3) is equivalent to

$$\begin{aligned} I(u) &:= \left| u^* \left(\frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j a_j^* - \frac{1}{2}(I_n - \frac{1}{2}e_1 e_1^*) \right) u \right| \\ &= \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) |a_j^* u|^2 - \frac{1}{2}(1 - \frac{1}{2}|u(1)|^2) \right| \\ &\leq \frac{\theta}{4} \end{aligned}$$

for any $u \in \mathbb{C}^n$ with $\|u\| = 1$. To this end, we write u in the form of $u = (u(1), \tilde{u})^*$ with $\tilde{u} = (u(2), \dots, u(n))^* \in \mathbb{C}^{n-1}$. Similarly, we write $a_j = (a_j(1), \tilde{a}_j)^*$. Then

$$|a_j^* u|^2 = |a_j(1)|^2 |u(1)|^2 + 2\operatorname{Re}(a_j(1) \overline{u(1)} (\tilde{a}_j^* \tilde{u})) + |\tilde{a}_j^* \tilde{u}|^2.$$

Hence, noting that $\|u\|^2 = |u(1)|^2 + \|\tilde{u}\|^2 = 1$, we obtain that

$$\begin{aligned}
I(u) &= \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) \left(|a_j(1)|^2 |u(1)|^2 + 2\operatorname{Re}(a_j(1)\overline{u(1)}(\tilde{a}_j^* \tilde{u})) + |\tilde{a}_j^* \tilde{u}|^2 \right) \right. \\
&\quad \left. - \frac{1}{2} \left(|u(1)|^2 + \|\tilde{u}\|^2 - \frac{1}{2}|u(1)|^2 \right) \right| \\
&= \left| \frac{1}{m} \sum_{j=1}^m \left(\exp(-|a_j(1)|^2) |a_j(1)|^2 |u(1)|^2 + 2\operatorname{Re}(\exp(-|a_j(1)|^2) a_j(1)\overline{u(1)} \tilde{a}_j^* \tilde{u}) + \exp(-|a_j(1)|^2) |\tilde{a}_j^* \tilde{u}|^2 \right) \right. \\
&\quad \left. - \frac{1}{2} \|\tilde{u}\|^2 - \frac{1}{4} |u(1)|^2 \right| \\
&\leq \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) |a_j(1)|^2 - \frac{1}{4} \right| \cdot |u(1)|^2 + \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) - \frac{1}{2} \right| \cdot \|\tilde{u}\|^2 \\
&\quad + \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) (|\tilde{a}_j^* \tilde{u}|^2 - \|\tilde{u}\|^2) \right| + 2 \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j(1)\overline{u(1)} \tilde{a}_j^* \tilde{u} \right| \\
(2.4) \quad &\leq 2\epsilon + \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) (|\tilde{a}_j^* \tilde{u}|^2 - \|\tilde{u}\|^2) \right| + 2 \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j(1)\overline{u(1)} \tilde{a}_j^* \tilde{u} \right|.
\end{aligned}$$

Here in the last inequality, as

$$(2.5) \quad 0 \leq \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) |a_j(1)|^2 \leq \frac{1}{e} \quad \text{and} \quad 0 \leq \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) \leq 1,$$

we use Hoeffding inequality to obtain that for any $\epsilon > 0$, there exists a constant C_1 , so that when $m \geq C_1 n$,

$$\left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) |a_j(1)|^2 - \frac{1}{4} \right| \leq \epsilon$$

and

$$\left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) - \frac{1}{2} \right| \leq \epsilon$$

hold with probability at least $1 - 2\exp(-2\gamma_\epsilon n)$.

We next consider the third term in (2.4), i.e.,

$$2 \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j(1)\overline{u(1)} \tilde{a}_j^* \tilde{u} \right|.$$

According to Hoeffding-type inequality [14, Proposition 5.10], when $m \geq C_2 \sqrt{n \sum_{j=1}^m \exp^2(-|a_j(1)|^2) |a_j(1)|^2}$,

$$2 \left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) a_j(1)\overline{u(1)} \tilde{a}_j^* \tilde{u} \right| \leq \theta_0$$

holds with probability at least $1 - 3\exp(-2\gamma_{\theta_0} n)$. Here, $\theta_0 > 0$ is an arbitrary constant. Next we consider the second term in (2.4), i.e.,

$$\left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) (|\tilde{a}_j^* \tilde{z}|^2 - \|\tilde{z}\|^2) \right|.$$

Using Bernstein-type inequality, when

$$m \geq C_3 \left(\sqrt{n \sum_{j=1}^m \exp^2(-|a_j(1)|^2)} + n \cdot \max_j \exp(-|a_j(1)|^2) \right),$$

with probability at least $1 - 2\exp(-2\gamma_{\theta_0}n)$, we have

$$\left| \frac{1}{m} \sum_{j=1}^m \exp(-|a_j(1)|^2) (|\tilde{a}_j^* \tilde{z}|^2 - \|\tilde{z}\|^2) \right| \leq \theta_0.$$

Therefore, for any unit norm vector u ,

$$I(u) \leq 2\epsilon + 2\theta_0$$

holds with probability at least $1 - 7\exp(-2\gamma_{\epsilon, \theta_0}n)$. By Lemma 5.4 in [14], we can bound the operator norm via an ϵ -net argument:

$$\max_u I(u) \leq 2 \max_{u \in \mathcal{N}} I(u) \leq 4\epsilon + 4\theta_0,$$

where \mathcal{N} is an $\frac{1}{4}$ -net of the unit sphere in \mathbb{C}^n . By applying the union bound and choosing appropriate θ_0 , ϵ and γ , (2.3) holds with probability at least $1 - 7\exp(-\gamma'_\theta n)$ as long as

$$\begin{aligned} m &\geq C_1 n + C_2 \sqrt{n \sum_{j=1}^m \exp^2(-|a_j(1)|^2) |a_j(1)|^2} \\ &\quad + C_3 \left(\sqrt{n \sum_{j=1}^m \exp^2(-|a_j(1)|^2)} + n \cdot \max_j \exp(-|a_j(1)|^2) \right) \\ (2.6) \quad &\geq C_1 n + C'_2 \sqrt{mn} + C'_3 (\sqrt{mn} + n), \end{aligned}$$

where the last inequality follows from (2.5).

We can choose a sufficiently large constant $C' \geq C_1 + C'_2 \sqrt{C'} + C'_3 (\sqrt{C'} + 1)$, so that (2.6) holds provided $m \geq C'n$. Thus when $m \geq C'n$, with probability at least $1 - 7\exp(-\gamma'_\theta n)$, (2.3) holds. \square

Now we begin to prove the Theorem 2.1.

Proof of Theorem 2.1. Suppose \tilde{x}_0 with $\|\tilde{x}_0\| = 1$ is the eigenvector corresponding to the minimum eigenvalue λ of

$$Y = \frac{1}{m} \sum_{j=1}^m \exp(-|a_j^* z|^2) a_j a_j^*.$$

Note that

$$\mathbb{E}(Y) = \frac{1}{2} (I_n - \frac{1}{2} z z^*)$$

and the minimum eigenvalue of $\mathbb{E}(Y)$ is $\frac{1}{4}$. Then from Lemma 2.2, for any $0 < \theta \leq 1/2$

$$\|Y - \mathbb{E}(Y)\| \leq \frac{\theta}{4}$$

holds with probability at least $1 - 7\exp(-\gamma'_\theta n)$ provided $m \geq C'n$. Next according to the Weyl Theorem, we have

$$(2.7) \quad |\lambda_{\min}(\mathbb{E}Y) - \lambda_{\min}(Y)| = \left| \frac{1}{4} - \lambda \right| \leq \frac{\theta}{4}.$$

On the other hand,

$$\begin{aligned} (2.8) \quad \frac{\theta}{4} &\geq \|Y - \frac{1}{2} (I_n - \frac{1}{2} z z^*)\| \\ &\geq |\tilde{x}_0^* (Y - \frac{1}{2} (I_n - \frac{1}{2} z z^*)) \tilde{x}_0| \\ &= \left| \lambda - \frac{1}{2} + \frac{1}{4} |\tilde{x}_0^* z|^2 \right| \\ &\geq \left| \frac{1}{4} |\tilde{x}_0^* z|^2 - \frac{1}{4} \right| - \left| \frac{1}{4} - \lambda \right|. \end{aligned}$$

Combining (2.7) and (2.8), we obtain

$$|\tilde{x}_0^* z|^2 \geq 1 - 2\theta.$$

So for any $0 < \theta \leq 1/2$, we have

$$\text{dist}^2(\tilde{x}_0, z) = \min_{\phi \in [0, 2\pi)} \|z - e^{i\phi} \tilde{x}_0\|^2 \leq \|z\|^2 + \|\tilde{x}_0\|^2 - 2|\tilde{x}_0^* z| \leq 2 - 2\sqrt{1 - 2\theta} \leq 5\theta.$$

Set $x_0 = \sqrt{\frac{1}{m} \sum_{j=1}^m |a_j^* z|^2} \tilde{x}_0$. By Lemma 2.1, when $m \geq c_\theta n$, with probability at least $1 - 2\exp(-\gamma n)$, we have

$$|\|x_0\| - 1|^2 \leq |\|x_0\|^2 - 1| = \left| \frac{1}{m} \sum_{j=1}^m |a_j^* z|^2 - 1 \right| \leq 5\theta.$$

Therefore,

$$\begin{aligned} \text{dist}(x_0, z) &\leq \|x_0 - \tilde{x}_0\| + \text{dist}(\tilde{x}_0, z) \\ &= |\|x_0\| - 1| + \text{dist}(\tilde{x}_0, z) \\ &\leq 2\sqrt{5\theta} \end{aligned}$$

holds with probability at least $1 - 9\exp(-\gamma_\theta n)$ provided $m \geq Cn$, where C is a sufficiently large constant. \square

Remark 2.1. In Algorithm 1, we take

$$Y = \frac{1}{m} \sum_{r=1}^m \exp(-y_r) a_r a_r^*.$$

It is possible to obtain similar results with replacing $\exp(-y_r)$ in Y by another bounded function $g(y_r)$. For example, we can take $g(y_r) = \exp(-y_r^p)$ where $0 < p \leq 1$.

Remark 2.2. Theorem 2.1 presents the performance of Algorithm 1 for the case $\mathbb{H} = \mathbb{C}$. In the real case, we set $Y := \frac{1}{m} \sum_{j=1}^m \exp(-|a_j^\top z|^2) a_j a_j^\top$ and then we have $\mathbb{E}(Y) = \frac{\sqrt{3}}{3}(I_n - \frac{2}{3}zz^\top)$. Using similar method with the proof of Theorem 2.1, we can obtain

$$\text{dist}^2(\tilde{x}_0, z) \leq 2 - 2\sqrt{1 - \frac{3\sqrt{3}}{4}\theta}$$

with probability at least $1 - 9\exp(-\gamma_\theta n)$. Then when $0 < \theta \leq \frac{3}{5}$, we have $\text{dist}(\tilde{x}_0, z) \leq \sqrt{2\theta}$. Hence, in the real case, when $0 < \theta \leq \frac{3}{5}$ and $m \geq Cn$, with probability at least $1 - 9\exp(-\gamma_\theta n)$, we have

$$\text{dist}(x_0, z) = \min\{\|x_0 - z\|, \|x_0 + z\|\} \leq 2\sqrt{2\theta}.$$

3. GAUSS-NEWTON METHOD FOR REAL PHASE RETRIEVAL

In this section, we consider the case where $\mathbb{H} = \mathbb{R}$. We formula (1.1) as a nonlinear least square problem

$$(3.9) \quad \min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - y_j)^2,$$

where $y_j = \langle a_j, z \rangle^2$. To state conveniently, we set $F_j(x) := \frac{1}{\sqrt{m}}(\langle a_j, x \rangle^2 - y_j)$ and then

$$f(x) = \frac{1}{2} \sum_{j=1}^m F_j(x)^2.$$

3.1. Gauss-Newton Method. To solve the real phase retrieval problem (1.1), we consider

$$(3.10) \quad \min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - y_j)^2 = \frac{1}{2} \sum_{j=1}^m F_j(x)^2,$$

where $F_j(x) = \frac{1}{\sqrt{m}}(\langle a_j, x \rangle^2 - y_j)$.

The (3.10) is a quadratic polynomial optimization problem. To solve it, our algorithm uses the well-known Gauss-Newton iteration. To make the paper self-contained, we introduce the Gauss-Newton iteration in detail. Starting from an initial guess x_0 , we refine the k -th iteration point x_k by the update rule:

$$(3.11) \quad x_{k+1} = x_k - \left(2 \sum_{j=1}^m \langle a_j, x_k \rangle^2 \cdot a_j a_j^\top \right)^{-1} \left(\sum_{j=1}^m (\langle a_j, x_k \rangle^2 - y_j) (a_j a_j^\top) x_k \right).$$

The idea behind the iteration is to replace $F_j(x)$ by its linear approximation at x_k :

$$\begin{aligned} F_j(x) &\approx F_j(x_k) + \nabla F_j(x_k)^\top (x - x_k) \\ &= \frac{1}{\sqrt{m}} (x_k^\top a_j a_j^\top x_k - y_j + 2(a_j a_j^\top x_k)^\top (x - x_k)). \end{aligned}$$

Suppose that $J(x_k) \in \mathbb{R}^{m \times n}$ and the j -th row of $J(x_k)$ is $\frac{2}{\sqrt{m}}(a_j a_j^\top x_k)^\top$. Suppose that the j -th component of $F(x_k) \in \mathbb{R}^m$ is given by $F_j(x_k)$, $j = 1, \dots, m$. Then the following least square problem can be considered as an approximation to (3.10):

$$(3.12) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} \|J(x_k)(x - x_k) + F(x_k)\|_2^2.$$

We choose the next iteration point x_{k+1} as the solution to (3.12), i.e.,

$$\begin{aligned} x_{k+1} &= x_k - (J(x_k)^\top J(x_k))^{-1} J(x_k) F(x_k) \\ &= x_k - (J(x_k)^\top J(x_k))^{-1} \nabla f(x_k) \\ &= x_k - \left(2 \sum_{j=1}^m \langle a_j, x_k \rangle^2 \cdot a_j a_j^\top \right)^{-1} \left(\sum_{j=1}^m (\langle a_j, x_k \rangle^2 - y_j) (a_j a_j^\top) x_k \right). \end{aligned}$$

The Gauss-Newton method uses Algorithm 1 to obtain an initial guess x_0 and iteratively refine x_k by the rule (3.11). As we need the current measurements are independent with the last iteration point, we re-sample A in every iteration. Then Algorithm 2 is in fact a variant of Gauss-Newton method with using different measurements in each iteration. The re-sampling idea is also used in [12] for the alternating minimization algorithm and in [7] for the WF algorithm with coded diffraction patterns.

Algorithm 2 Gauss-Newton Method with Re-sampling

Input: Measurement matrix: $A \in \mathbb{R}^{m \times n}$, observations: $y \in \mathbb{R}^m$ and the number of iterations k_0

- 1: Partition y and the corresponding rows of A into $k_0 + 1$ equal disjoint sets: $(y^{(0)}, A^{(0)}), (y^{(1)}, A^{(1)}), \dots, (y^{(k_0)}, A^{(k_0)})$. The number of rows in $A^{(j)}$ is $m' = m/(k_0 + 1)$.
- 2: Set

$$\lambda := \sqrt{\frac{1}{m'} \sum_j y_j^{(0)}}.$$

Set x_0 to be the minimum eigenvector of $\sum_j \exp(-y_j^{(0)}) a_j^{(0)} a_j^{(0)\top}$ and $\|x_0\| = \lambda$.

- 3: For $k = 0, 1, \dots, k_0 - 1$ do

$$\begin{aligned} x_{k+1} &= x_k - (J^{k+1}(x_k) J^{k+1}(x_k))^{-1} \nabla f^{k+1}(x_k) \\ &= x_k - \left(2 \sum_{j=1}^{m'} \langle a_j, x_k \rangle^2 \cdot a_j a_j^\top \right)^{-1} \left(\sum_{j=1}^{m'} (\langle a_j, x_k \rangle^2 - y_j) (a_j a_j^\top) x_k \right), \end{aligned}$$

where $y_1, \dots, y_{m'}$ are the entries of $y^{(k+1)}$ and $a_1, \dots, a_{m'}$ are the rows of $A^{(k+1)}$.

- 4: End for

Output: x_{k_0} .

3.2. Convergence of Gauss-Newton Method with Re-sampling. We study the performance of Gauss-Newton method with re-sampling in this subsection. Theorem 3.1 illustrates that under given conditions, Algorithm 2 has a quadratic convergence rate. Theorem 3.2 shows that to achieve an ϵ accuracy, the Gauss-Newton method needs $\mathcal{O}(\log \log(\frac{1}{\epsilon}))$ iterations.

Theorem 3.1. *Let $z \in \mathbb{R}^n$ be an arbitrary vector with $\|z\| = 1$ and $y_j = |\langle a_j, z \rangle|^2$, where $a_j \in \mathbb{R}^n$, $j = 1, \dots, m$ are Gaussian random measurements with $m \geq Cn \log n$. Suppose $0 < \delta \leq 1/81$ is a constant and $x_k \in \mathbb{R}^n$ satisfying $\text{dist}(x_k, z) \leq \sqrt{\delta}$. Suppose that x_{k+1} is defined by (3.11). With probability at least $1 - c/n^2$, we have*

$$(3.13) \quad \text{dist}(x_{k+1}, z) \leq \beta \cdot \text{dist}^2(x_k, z),$$

where

$$(3.14) \quad \beta = \frac{24(4 + \frac{\delta}{2})(1 + \sqrt{\delta})}{(16 - \delta)(1 - \sqrt{\delta})^2}.$$

Remark 3.1. *In Theorem 3.1, the reason why we require $0 < \delta \leq 1/81$ is to guarantee $\beta \cdot \delta \leq \sqrt{\delta}$. Hence the condition $\text{dist}(x_{k+1}, z) \leq \beta \cdot \delta \leq \sqrt{\delta}$ still holds and we can use Theorem 3.1 at the $(k+1)$ -th iteration.*

According to Remark 2.2, for any $0 < \delta \leq 1/81$, $0 < \theta \leq \delta/8$, we have

$$\text{dist}(x_0, z) = \min\{\|x_0 - z\|, \|x_0 + z\|\} \leq 2\sqrt{2\theta} \leq \sqrt{\delta}$$

with probability at least $1 - 9\exp(-\gamma_\theta n)$ provided $m \geq Cn$. Combining this initialization result with Theorem 3.1, we have the following conclusion.

Theorem 3.2. *Suppose that $z \in \mathbb{R}^n$ with $\|z\| = 1$ is an arbitrary vector and $a_j \in \mathbb{R}^n$, $j = 1, \dots, m$ are Gaussian random measurements. Suppose that ϵ is an arbitrary constant within range $(0, 1/2)$ and $\delta \in (0, 1/81]$ is a fixed constant. Set $y = Az$ and $k_0 \geq \max\{0, \log_2 \log_2 \frac{1}{\epsilon} - \log_2 \log_2 \frac{1}{\beta\sqrt{\delta}}\}$ and β is given in (3.14). If $m \geq C \cdot \log_2 \log_2 \frac{1}{\epsilon} \cdot n \log n$, then with probability at least $1 - \tilde{c}/n^2$, Algorithm 2 outputs x_{k_0} such that*

$$\text{dist}(x_{k_0}, z) < \epsilon,$$

where C is a constant depending on δ .

Proof. Suppose that $0 < \delta \leq 1/81$. According to the real version of Theorem 2.1 (see Remark 2.2), we have

$$\text{dist}(x_0, z) \leq \sqrt{\delta}$$

with probability at least $1 - 9\exp(-\gamma_\delta n)$. According to the discussion in Remark 3.1, we have

$$\beta \cdot \delta \leq \sqrt{\delta},$$

where β is defined in Theorem 3.1. Iterating (3.13) in Theorem 3.1 k_0 times leads to

$$\begin{aligned} \text{dist}(x_{k_0}, z) &\leq \beta \cdot \text{dist}^2(x_{k_0-1}, z) \\ &\leq \beta^{2^{k_0}-1} \text{dist}^{2^{k_0}}(x_0, z) \\ &\leq \beta^{2^{k_0}-1} \cdot (\sqrt{\delta})^{2^{k_0}} \\ &\leq (\beta \cdot \sqrt{\delta})^{2^{k_0}} \\ &\leq \epsilon, \end{aligned}$$

which holds with probability at least $1 - \tilde{c}/n^2$. \square

3.3. Proof of Theorem 3.1. In this section, we devote to prove the Theorem 3.1. At first, we give some essential lemmas.

Lemma 3.1. [Lemma 7.4 in [7]] Suppose that a_j , $j = 1, 2, \dots, m$ are Gaussian random measurements and $m \geq Cn \log n$, where C is sufficiently large. Set

$$S := \frac{1}{m} \sum_{j=1}^m |a_j(1)|^2 a_j a_j^*.$$

Then for any $\delta > 0$,

$$\|S - \mathbb{E}(S)\| \leq \frac{\delta}{4}$$

holds with probability at least $1 - 5\exp(-\gamma_\delta n) - 4/n^2$.

Recall that $S_k := \{\lambda z + (1 - \lambda)x_k : 0 \leq \lambda \leq 1\}$, $J(x) := (\nabla F_1(x), \dots, \nabla F_m(x))^\top$ and

$$J(x)^\top J(x) = \sum_{j=1}^m \nabla F_j(x) \nabla F_j(x)^\top = \frac{4}{m} \sum_{j=1}^m \langle a_j, x \rangle^2 \cdot a_j a_j^\top.$$

We set

$$\begin{aligned} (3.15) \quad H(x) &:= \nabla^2 f(x) - J(x)^\top J(x) \\ &= \frac{2}{m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - y_j) a_j a_j^\top. \end{aligned}$$

Lemma 3.2. Suppose that $x_k \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ ($\|z\| = 1$) and $\|x_k - z\| \leq \sqrt{\delta}$, where $0 < \delta \leq 1/81$ is a constant. Suppose that the measurement vectors a_j , $j = 1, \dots, m$ are Gaussian random measurements which are independent with x_k and z . Then when $m \geq Cn \log n$,

$$J(x)^\top J(x) = \frac{4}{m} \sum_{j=1}^m \langle a_j, x \rangle^2 a_j a_j^\top$$

is L_J -Lipschitz continuous on S_k with probability at least $1 - 5\exp(-\gamma_\delta n) - 4/n^2$, i.e., for any $x, y \in S_k$,

$$\|J(x)^\top J(x) - J(y)^\top J(y)\| \leq L_J \|x - y\|,$$

where $L_J = 8(4 + \frac{\delta}{2})(1 + \sqrt{\delta})$.

Proof. Since the measurement vectors a_j , $j = 1, \dots, m$ are rotationally invariant and independent with x_k and z , wlog, we can assume that $z = e_1$ and $x_k = \|x_k\|(\alpha e_1 + \sqrt{1 - \alpha^2} e_2)$, where $\alpha = \langle x_k, z \rangle / \|x_k\|$. As $\|x_k - z\| \leq \sqrt{\delta}$, so $\langle x_k, z \rangle \geq 0$, i.e., $\alpha \geq 0$. We can write $x, y \in S_k$ in the form of

$$\begin{cases} x = \lambda_1 x_k + (1 - \lambda_1)z, & \lambda_1 \in [0, 1], \\ y = \lambda_2 x_k + (1 - \lambda_2)z, & \lambda_2 \in [0, 1]. \end{cases}$$

For any $x, y \in S_k$, we have

$$\begin{aligned}
\|J(x)^\top J(x) - J(y)^\top J(y)\| &= 4 \left\| \frac{1}{m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - \langle a_j, y \rangle^2) a_j a_j^\top \right\| \\
&= 4 \left\| \frac{1}{m} \sum_{j=1}^m \langle a_j, x+y \rangle \langle a_j, x-y \rangle a_j a_j^\top \right\| \\
(3.16) \quad &\leq 4 \|x+y\| \|x-y\| \left\| \frac{1}{m} \sum_{j=1}^m ((a_j^\top e_1)^2 + (a_j^\top e_2)^2) a_j a_j^\top \right\|,
\end{aligned}$$

where the last inequality is obtained by Cauchy-Schwarz inequality. Next we set

$$S := \frac{1}{m} \sum_{j=1}^m ((a_j^\top e_1)^2 + (a_j^\top e_2)^2) a_j a_j^\top.$$

Then by calculation, we have $\mathbb{E}S = 2I_n + 2e_1 e_1^\top + 2e_2 e_2^\top$ and $\|\mathbb{E}S\| = 4$. According to Lemma 3.1, for $0 < \delta \leq 1/81$ and $m \geq Cn \log n$,

$$\|S - \mathbb{E}S\| \leq \frac{\delta}{2}$$

holds with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$. So

$$(3.17) \quad \|S\| \leq 4 + \frac{\delta}{2}.$$

On the other hand, as $\|x_k - z\| \leq \sqrt{\delta}$, we have

$$1 - \sqrt{\delta} \leq \|x_k\| \leq 1 + \sqrt{\delta}.$$

Thus

$$\begin{aligned}
(3.18) \quad \|x+y\| &= \|(\lambda_1 + \lambda_2)x_k + (2 - \lambda_1 - \lambda_2)z\| \leq (\lambda_1 + \lambda_2)\|x_k\| + (2 - \lambda_1 - \lambda_2) \\
&\leq 2(1 + \sqrt{\delta}).
\end{aligned}$$

Putting (3.17) and (3.18) into (3.16), we obtain

$$\|J(x)^\top J(x) - J(y)^\top J(y)\| \leq 8(4 + \frac{\delta}{2})(1 + \sqrt{\delta})\|x - y\|.$$

So we conclude that when $m \geq Cn \log n$, $J(x)^\top J(x)$ is Lipschitz continuous on the line S_k with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$. \square

Corollary 3.1. *Under the same conditions as in Lemma 3.2,*

$$H(x) = \frac{2}{m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - y_j) a_j a_j^\top$$

is Lipschitz continuous on S_k with Lipschitz constant

$$L_H = \frac{1}{2} L_J = 4 \cdot \left(4 + \frac{\delta}{2}\right) \cdot (1 + \sqrt{\delta}),$$

with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$.

Proof. According to Lemma 3.2, we have $J(x)^\top J(x)$ is L_J -Lipschitz continuous on S_k . That is to say,

$$\text{for any } x, y \in S_k, \quad \|J(x)^\top J(x) - J(y)^\top J(y)\| \leq L_J \|x - y\|.$$

While for any $x, y \in S_k$,

$$\begin{aligned} H(x) - H(y) &= \frac{2}{m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - y_j - \langle a_j, y \rangle^2 + y_j) a_j a_j^\top \\ &= \frac{2}{m} \sum_{j=1}^m (\langle a_j, x \rangle^2 - \langle a_j, y \rangle^2) a_j a_j^\top \\ &= \frac{1}{2} (J(x)^\top J(x) - J(y)^\top J(y)). \end{aligned}$$

Then

$$\begin{aligned} \|H(x) - H(y)\| &= \left\| \frac{1}{2} (J(x)^\top J(x) - J(y)^\top J(y)) \right\| \\ &\leq \frac{1}{2} L_J \|x - y\|. \end{aligned}$$

So $H(x)$ is Lipschitz continuous on S_k with constant $L_H = \frac{1}{2} L_J$. \square

Next we present an estimation of the largest eigenvalue of $(J(x_k)^\top J(x_k))^{-1}$.

Lemma 3.3. *Suppose that $\|x_k - z\| \leq \sqrt{\delta}$ where $x_k, z \in \mathbb{R}^n$ with $\|z\| = 1$ and $0 < \delta \leq 1/81$. Suppose that $a_j, j = 1, \dots, m$ are Gaussian random measurements which are independent with x_k . If $m \geq Cn \log n$ for a sufficiently large constant C , then with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$, we have*

$$\|(J(x_k)^\top J(x_k))^{-1}\| \leq \frac{4}{(16 - \delta)(1 - \sqrt{\delta})^2}.$$

Proof. Set

$$S := J(x_k)^\top J(x_k) = \frac{4}{m} \sum_{j=1}^m \langle a_j, x_k \rangle^2 a_j a_j^\top.$$

After a simple calculation, we obtain

$$\mathbb{E}S = 4(\|x_k\|^2 I_n + 2x_k x_k^\top)$$

and the minimum eigenvalue of $\mathbb{E}S$ is

$$\lambda_{\min}(\mathbb{E}S) = 4\|x_k\|^2.$$

According to Lemma 3.1, for $0 < \delta \leq 1/81$ and $m \geq Cn \log n$,

$$\|S - \mathbb{E}S\| \leq \frac{\delta}{4} \|x_k\|^2$$

holds with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$. Then according to the Wely Theorem, we have

$$|\lambda_{\min}(S) - \lambda_{\min}(\mathbb{E}S)| \leq \|S - \mathbb{E}S\| \leq \frac{\delta}{4} \|x_k\|^2,$$

which implies that

$$\begin{aligned} \lambda_{\min}(S) &\geq (4 - \frac{\delta}{4}) \|x_k\|^2 \\ &\geq (4 - \frac{\delta}{4})(1 - \sqrt{\delta})^2. \end{aligned}$$

Here, we use (3.3) in the last inequality. Then with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$, we have

$$(3.19) \quad \lambda_{\max}(S^{-1}) = 1/\lambda_{\min}(S) \leq \frac{4}{(16 - \delta)(1 - \sqrt{\delta})^2},$$

which implies the conclusion. \square

We next present the proof of Theorem 3.1.

Proof of Theorem 3.1. Without loss of generality, we only consider the case where $\langle x_k, z \rangle \geq 0$, i.e.,

$$\text{dist}(x_k, z) = \|x_k - z\|.$$

Then we just need to prove when $\|x_k - z\| \leq \sqrt{\delta}$ and $m \geq Cn \log n$,

$$\text{dist}(x_{k+1}, z) = \|x_{k+1} - z\| \leq \beta \cdot \|x_k - z\|^2 = \beta \cdot \text{dist}^2(x_k, z)$$

holds with probability at least $1 - c/n^2$.

As z is an exact solution to (3.10), we have $\nabla f(z) = H(z) = 0$. The definition of x_{k+1} shows that

$$\begin{aligned} x_{k+1} - z &= x_k - z - (J(x_k)^\top J(x_k))^{-1} \nabla f(x_k) \\ (3.20) \quad &= (J(x_k)^\top J(x_k))^{-1} \left[(J(x_k)^\top J(x_k)) \cdot (x_k - z) - (\nabla f(x_k) - \nabla f(z)) \right]. \end{aligned}$$

Define $S_k := \{x_k + t(z - x_k) : 0 \leq t \leq 1\}$ and $x(t) = x_k + t(z - x_k)$. Then we have

$$\begin{aligned} \nabla f(x_k) - \nabla f(z) &= \nabla f(x(0)) - \nabla f(x(1)) \\ &= - \int_0^1 \frac{d(\nabla f(x(t)))}{dt} dt \\ &= - \int_0^1 \nabla^2 f(x(t)) \cdot x'(t) dt \\ (3.21) \quad &= - \frac{1}{\|x_k - z\|} \int_{S_k} \nabla^2 f(x) \cdot (z - x_k) ds \end{aligned}$$

The integral in (3.21) is interpreted as element-wise. Combining (3.15) and $H(z) = 0$, we obtain that

$$\begin{aligned} &\left\| (J(x_k)^\top J(x_k)) \cdot (x_k - z) - (\nabla f(x_k) - \nabla f(z)) \right\| \\ &= \frac{1}{\|x_k - z\|} \left\| \int_{S_k} (J(x_k)^\top J(x_k) \cdot (x_k - z) - \nabla^2 f(x) \cdot (x_k - z)) ds \right\| \\ (3.22) \quad &= \frac{1}{\|x_k - z\|} \left\| \int_{S_k} (J(x_k)^\top J(x_k) - J(x)^\top J(x) - H(x)) \cdot (x_k - z) ds \right\| \\ &\leq \frac{1}{\|x_k - z\|} \left(\left\| \int_{S_k} (J(x_k)^\top J(x_k) - J(x)^\top J(x)) \cdot (x_k - z) ds \right\| \right. \\ &\quad \left. + \left\| \int_{S_k} (H(x) - H(z)) \cdot (x_k - z) ds \right\| \right). \end{aligned}$$

According to Lemma 3.2 and Corollary 3.1, $J(x)^\top J(x)$ and $H(x)$ are Lipschitz continuous on the line S_k with probability at least $1 - 5 \exp(-\gamma_\delta n) - 4/n^2$ provided $m \geq Cn \log n$. So using (3.22), we obtain

$$\begin{aligned} &\left\| (J(x_k)^\top J(x_k)) \cdot (x_k - z) - (\nabla f(x_k) - \nabla f(z)) \right\| \\ &\leq \frac{1}{\|x_k - z\|} \left(\left\| \int_{S_k} (J(x_k)^\top J(x_k) - J(x)^\top J(x)) \cdot (x_k - z) ds \right\| + \left\| \int_{S_k} (H(x) - H(z)) \cdot (x_k - z) ds \right\| \right) \\ &\leq \int_{S_k} \|J(x_k)^\top J(x_k) - J(x)^\top J(x)\| ds + \int_{S_k} \|H(x) - H(z)\| ds \\ &\leq \int_{S_k} L_J \|x_k - x\| ds + \int_{S_k} L_H \|x - z\| ds \\ &= L_J \|x_k - z\| \int_0^1 t \cdot \|x_k - z\| dt + L_H \|x_k - z\| \int_0^1 (1 - t) \cdot \|x_k - z\| dt \\ &= \frac{L_J + L_H}{2} \cdot \|x_k - z\|^2 \\ &= 6(4 + \frac{\delta}{2})(1 + \sqrt{\delta}) \|x_k - z\|^2. \end{aligned}$$

Thus according to Lemma 3.3 and (3.20), when $m \geq Cn \log n$,

$$\begin{aligned}
 \|x_{k+1} - z\| &= \|(J(x_k)^\top J(x_k))^{-1}\| \cdot \|(J(x_k)^\top J(x_k)) \cdot (x_k - z) - (\nabla f(x_k) - \nabla f(z))\| \\
 (3.23) \quad &\leq \frac{4}{(16 - \delta)(1 - \sqrt{\delta})^2} \cdot 6(4 + \frac{\delta}{2})(1 + \sqrt{\delta}) \|x_k - z\|^2 \\
 &= \beta \cdot \|x_k - z\|^2
 \end{aligned}$$

holds with probability at least $1 - c/n^2$. Based on the discussion in Remark 3.1, we have

$$\|x_{k+1} - z\| \leq \beta \cdot \|x_k - z\|^2 \leq \sqrt{\delta}.$$

Then we have $\langle x_{k+1}, z \rangle \geq 0$, i.e., $\text{dist}(x_{k+1}, z) = \|x_{k+1} - z\|$. Then (3.23) implies the conclusion. \square

4. NUMERICAL EXPERIMENTS

The purpose of numerical experiments is to compare the performance of Gauss-Newton method with that of other existing methods as mentioned before. In our numerical experiments, the measurement matrix $A \in \mathbb{H}^{m \times n}$ is generated by Gaussian random matrix and the entries of original signal $z \in \mathbb{H}^n$ is drawn from standard normal distribution.

Example 4.1. In this example, we test the Algorithm 1 to compare the initial guess of Algorithm 1 with that of spectral initialization (SI) and truncated spectral initialization (TSI). We take $n = 128$ and change m within the range $[2n, 20n]$ with the step n in Figure 1(a), which takes $\mathbb{H} = \mathbb{R}$. In Figure 1(b), we take $\mathbb{H} = \mathbb{C}$ and change m within the range $[4n, 22n]$ with the step n . For each m , we repeat the experiment 50 times and record the average value of the relative error $\text{dist}(x_0, z)/\|z\|$. Figure 1 depicts that Algorithm 1 outperforms spectral initialization and truncated spectral initialization significantly.

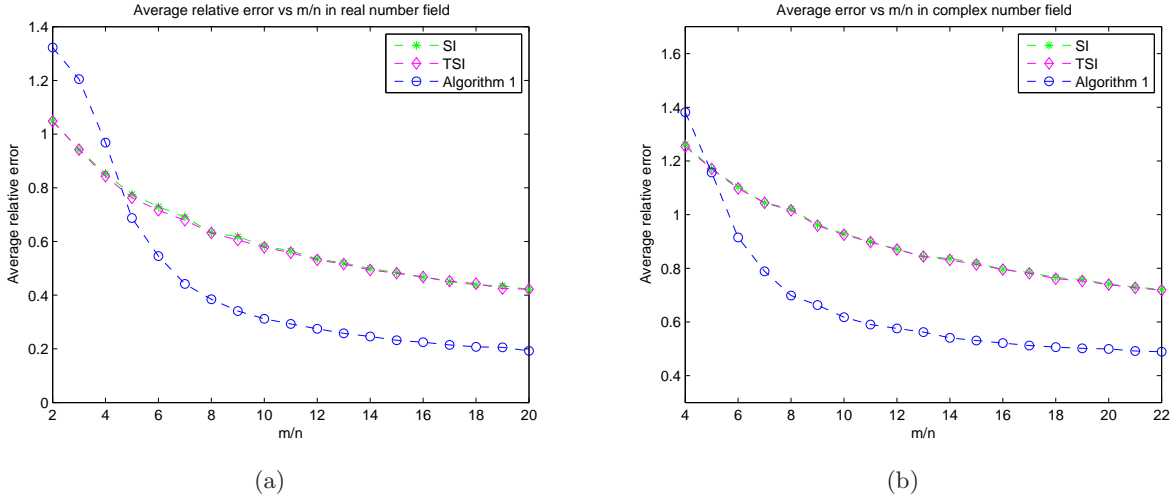


FIGURE 1. Initialization experiments: Averaged relative error between x_0 and z for $n = 128$ and (a) m/n changing within the range $[2, 20]$ with the step 1 in the real number field, and (b) m/n changing within the range $[4, 22]$ in the complex number field. The figures show that SI and TSI have similar performance in terms of average relative error, while Algorithm 1 performs better than the other two.

Example 4.2. We compare the performance of Gauss-Newton method (Algorithm 2) with that of WF method [7] and Altmin Phase method [12]. Both WF and Altmin Phase use SI for initialization. In Figure 2, we take $n = 128$ and $m/n = 4.5$. Figure 2 depicts the relative error against the iteration number. The numerical results show that Gauss-Newton method converges faster than WF method and Altmin Phase method.

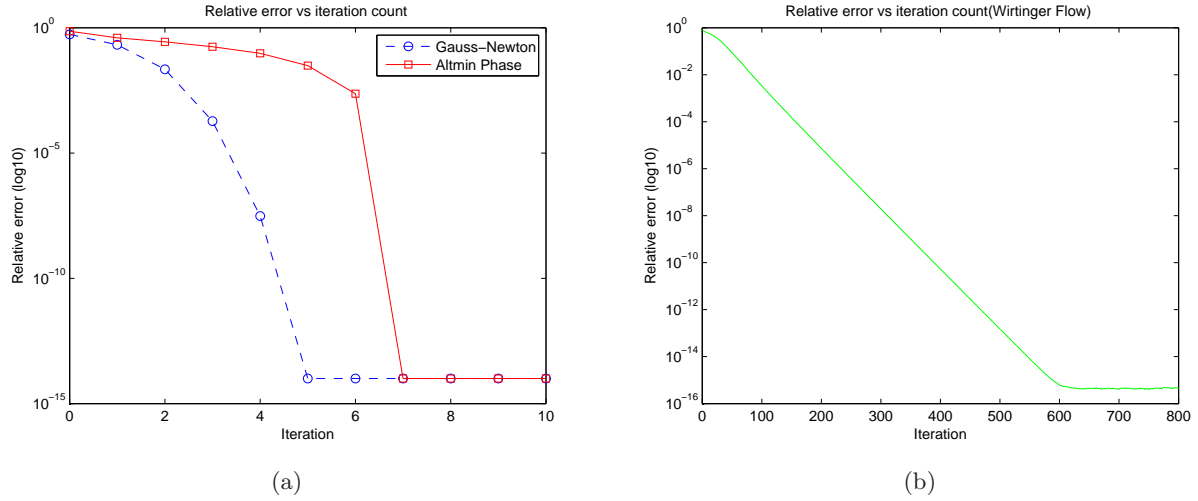


FIGURE 2. Performance experiments: Plot of relative error ($\log(10)$) vs number of iterations for (a) Gauss-Newton method and Altmin Phase method, and (b) WF method. The measurement matrix A is generated by real Gaussian random matrix with $n = 128$ and $m = 4.5n$. The figures show that Gauss-Newton method converges faster than WF method and Altmin Phase method.

Example 4.3. In this example, we test the success rate of Gauss-Newton method. Take $n = 128$ and change m/n within the range $[1, 6]$ with the step 0.5. For each m/n , we repeat 100 times and calculate the success rate. Figure 3 shows the numerical results with using the recovery algorithm Gauss-Newton, WF and Altmin Phase, respectively. The numerical results show that Gauss-Newton method has the better performance.

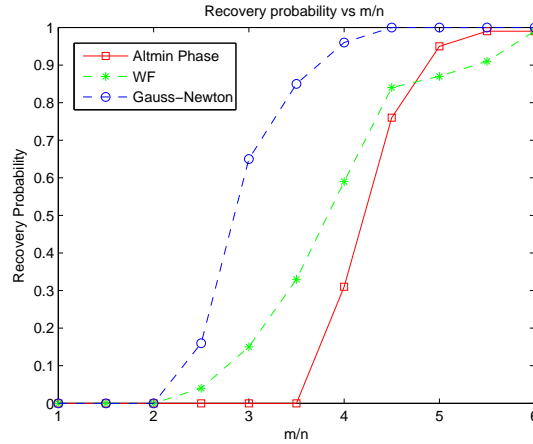


FIGURE 3. Success rate experiments: Empirical probability of successful recovery based on 100 random trails for different m/n . Take $n = 128$ and change m/n between 1 and 6. The figure demonstrates that Gauss-Newton method is better than WF method and Altmin Phase method in terms of success rate.

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